ISSN 0005-1179 (print), ISSN 1608-3032 (online), Automation and Remote Control, 2024, Vol. 85, No. 4, pp. 351–372.

On the External Estimation of Reachable and Null-Controllable Limit Sets for Linear Discrete-Time Systems with a Summary Constraint on the Scalar Control

D. N. Ibragimov

Moscow Aviation Institute (National Research University), Moscow, Russia
e-mail: rikk.dan@gmail.ru

Received October 29, 2023
Revised April 4, 2024
Accepted April 11, 2024

Abstract—The problem of constructing reachable and null-controllable sets for stationary linear discrete-time systems with a summary constraint on the scalar control is considered. For the case of quadratic constraints and a diagonalizable matrix of the system, these sets are built explicitly in the form of ellipsoids. In the general case, the limit reachable and null-controllable sets are represented as fixed points of a contraction mapping in the metric space of compact sets. On the basis of the method of simple iteration, a convergent procedure for constructing their external estimates with an indication of the a priori approximation error is proposed. Examples are given.

Keywords: linear discrete-time system, controllable limit set, reachable limit set, Hausdorff distance, contraction mapping principle

DOI: 10.31857/S0005117924040016

1. INTRODUCTION

It is often necessary to take into account various constraints on control actions when studying dynamic systems, which leads to the unreachability of some terminal states from a given initial state even in infinite time. As a result, the classical Kalman controllability conditions are not sufficient to draw a conclusion about the reachability of one or another terminal state. In this regard, it seems relevant to develop methods for constructing a constructive description of reachable sets, i.e. sets of terminal states to which the system can be transferred from the origin, and null-controllable sets, i.e. sets of initial states from which the system can be transferred to the origin, in a finite number of steps, as well as estimates of the limit reachable and null-controllable sets [1]. The null-controllable and reachable sets can be used in optimal control problems to construct positional control for discrete-time systems [2, 3]. Thus it is possible to judge the solvability of these problems in principle by the apparatus of limit sets.

At the moment, methods for estimating reachable sets of various classes of discrete-time systems [4], hybrid systems [5] and systems with various types of uncertainties [6] are being actively developed. Analytical representations of reachable and null-controllable sets for linear systems with discrete time and constraints on the control function in the sense of the l_∞-norm are known. In particular, it is proved that in the case of linear constraints on the control, the reachable and null-controllable sets in a finite number of steps are polyhedra [2]. For their limit analogues, necessary and sufficient conditions for the boundedness are formulated [7–9]. At the same time, most of the papers are either focused on studying only the general properties of reachable and null-controllable limit sets [8–12], or consider systems with unbounded control [10–14]. Only in a few special cases constructive methods for constructing external estimates have been proposed based on the apparatus of support half-spaces [15, 16] or the maximum principle [17].

For systems with summary control constraints, a description of the reachable and null-controllable limit sets in the form of polytopes is obtained for the case of constraints in the sense of the l_1-norm [18]. When choosing a l_p-norm with an arbitrary value of the parameter $p \in (1; +\infty)$, the general properties of reachable and null-controllable limit sets have been formulated and proven [19]. In particular, their representation in the form of projections of superellipsoidal sets of finite [20, 21] and infinite dimensions has been constructed, which is closely related to strictly convex analysis [22, 23], convex programming [24], and the theory of normed spaces [25] and linear operators [26].

Often in control problems it is necessary to examine a given initial state for reachability and controllability, which reduces to checking whether a some point in the phase space belongs to the reachable or controllable limit set. This procedure can be reduced to calculating the Minkowski functional, but the known results of [19] are not enough to construct it explicitly. Moreover, describing the Minkowski functional of the image of a convex set under a linear mapping in the general case is a non-trivial task. For this reason, it is relevant to develop methods implemented in software that will allow us to calculate exactly the Minkowski functional of the reachable and null-controllable limit sets or their external estimates of an arbitrary accuracy.

The paper studies the issues of constructing the Minkowski functional of reachable and null-controllable sets with a summary control constraint in the sense of the l_p-norm in the case when they are bounded. It is possible to explicitly describe this function under quadratic restrictions on the control and prove that the sets under study are ellipsoids. For the case of arbitrary normed spaces, the reachable and null-controllable limit sets are described as a fixed-point of a contraction mapping in the space of compacta with the Hausdorff distance. This allows us to propose a convergent iterative process for constructing external estimates of these sets with an explicit form of the a priori error. For a number of parameter values, the resulting estimates have a polyhedral structure, which makes it possible to use them in computer calculations.

The contents of the article are as follows. In Section 2 the problem is stated. Section 3 discusses the issues of calculating the Minkowski functional of reachable and null-controllable limit sets. For the special case of l_2-constraints on control and the diagonalizable matrix of the system, the corresponding sets are constructed explicitly. Section 4 describes the apparatus of contraction mappings used to construct reachable and null-controllable limit sets. Section 5 suggests a method for constructing external estimates of these sets of an arbitrary accuracy by the simple iteration method. Section 6 demonstrates the effectiveness of the developed mathematical apparatus through various examples.

2. PROBLEM STATEMENT

We consider a linear discrete-time system with summary constraint on scalar control:

$$
\begin{aligned}
&x(k+1) = Ax(k) + bu(k), \ k \in \mathbb{N} \cup \{0\}, \\
&x(0) = x_0, \ \sum_{k=0}^{\infty} |u(k)|^p \leqslant 1,
\end{aligned}
\tag{1}
$$

where $x(k) \in \mathbb{R}^n$ is the state vector of the system, $u(k) \in \mathbb{R}$ is the scalar control action, $A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$ are system matrices, $p > 1$ is a parameter that determines the type of summary control constraint.

For an arbitrary $N \in \mathbb{N} \cup \{0\}$ we denote by $\mathcal{Y}_p(N)$ the reachable set of the system (1), i.e. the set of states to which the system (1) can be transferred in N steps from the origin by an admissible control:

$$\mathcal{Y}_p(N) = \begin{cases} \left\{ x \in \mathbb{R}^n \colon x = \sum_{k=0}^{N-1} A^{N-k-1} b u(k), \ \sum_{k=0}^{N-1} |u(k)|^p \leqslant 1 \right\}, & N \in \mathbb{N}, \\ \{0\}, & N = 0. \end{cases} \tag{2}$$

We denote by $\mathcal{Y}_{p,\infty}$ the reachable limit set of the system (1), i.e. the set of states to which the system (1) can be transferred in any finite number of steps by an admissible control:

$$\mathcal{Y}_{p,\infty} = \bigcup_{N=0}^{\infty} \mathcal{Y}_p(N). \tag{3}$$

For an arbitrary $N \in \mathbb{N} \cup \{0\}$ we denote by $\mathcal{X}_p(N)$ the null-controllable set of the system (1), i.e. the set of initial states from which the system (1) can be transferred to the origin in N steps by an admissible control:

$$\mathcal{X}_p(N) = \begin{cases} \left\{ x_0 \in \mathbb{R}^n \colon -A^N x_0 = \sum_{k=0}^{N-1} A^{N-k-1} b u(k), \ \sum_{k=0}^{N-1} |u(k)|^p \leqslant 1 \right\}, & N \in \mathbb{N}, \\ \{0\}, & N = 0. \end{cases} \tag{4}$$

We denote by $\mathcal{X}_{p,\infty}$ the set null-controllable limit set of the system (1), i.e. the set of initial states from which the system (1) can be transferred to the origin in any finite number of steps by an admissible control:

$$\mathcal{X}_{p,\infty} = \bigcup_{N=0}^{\infty} \mathcal{X}_p(N). \tag{5}$$

It is required to develop an effective method for constructing an external estimate of the sets (3) and (5) with any predetermined accuracy. The Hausdorff distance ρ_H is considered as an accuracy criterion, and all sets are assumed to be elements of the complete metric space $(\mathbb{K}_n, \rho_H)$ [27]:

$$\mathbb{K}_n = \{\mathcal{X} \subset \mathbb{R}^n \colon \mathcal{X} \text{ is compact}\},$$

$$\rho_H(\mathcal{X}, \mathcal{Y}) = \max \left\{ \sup_{x \in \mathcal{X}} \inf_{y \in \mathcal{Y}} \|x - y\|_r ; \sup_{y \in \mathcal{Y}} \inf_{x \in \mathcal{X}} \|x - y\|_r \right\},$$

$$\|x\|_r = \begin{cases} \left(\sum_{i=1}^{n} |x_i|^r \right)^{\frac{1}{r}}, & r \geqslant 1, \\ \max_{i=\overline{1,n}} |x_i|, & r = \infty. \end{cases}$$

3. THE PROBLEM OF AN EXACT DESCRIPTION OF THE REACHABLE AND 0-CONTROLLABLE LIMIT SETS

Let us denote by $\mathcal{E}_p(\infty)$ a unit ball in the normed space l_p [25]:

$$\mathcal{E}_p(\infty) = \left\{ u \in l_p \colon \sum_{k=1}^{\infty} |u_k|^p \leqslant 1 \right\}.$$

We also identify the sequence $B = (b_1, b_2, \ldots) \in l_q^n$ with the linear operator $B \colon l_p \to \mathbb{R}_r^n$, acting according to the following rule:

$$Bu = \sum_{k=1}^{\infty} u_k b_k.$$

It is assumed here that the numbers p and q satisfy the relation $\frac{1}{p} + \frac{1}{q} = 1$, and the space $\mathbb{R}^n_r$ is a normed space $(\mathbb{R}^n, \|\cdot\|_r)$. Hence, taking into account the Riesz theorem [25], it follows that the operator B is bounded, which allows us to consider it as an element of the normed space l^n_q with an operator norm defined on it:

$$\|B\|_{l^n_q} = \sup_{u \in \mathcal{E}_p(\infty)} \|Bu\|_r.$$

For simplicity, we also identify an arbitrary sequence $y \in l_q$ with the linear and bounded functional $y\colon l_p \to \mathbb{R}$ generated by it according to the Riesz theorem:

$$(y, u) = \sum_{k=1}^{\infty} y_k u_k.$$

The necessary and sufficient conditions for the boundedness of the sets (3) and (5), determined by the system matrices A and b, are essential. The Jordan basis of the matrix A is a set of linearly independent vectors $h_1, \ldots, h_n \subset \mathbb{R}^n$, which specifies the similarity transformation of the matrix A to its real Jordan canonical form [28, Section 3.4 Ch. 3]. Such a basis is unique up to non-zero factors and to the order of the vectors $h_1, \ldots, h_n$, and each basis vector corresponds to some Jordan cell, i.e. to some eigenvalue of the matrix A. If we divide the elements of the Jordan basis into three sets according to the criterion of whether they correspond to an eigenvalue of the matrix A greater than, equal to, or less than 1 in modulus, then we can determine the following three invariant subspaces:

$$\begin{aligned}
\mathbb{L}_{<1} &= \mathrm{Lin}\{h_i\colon h_i \text{ correspond to eigenvalue } \lambda,\ |\lambda| < 1\},\\
\mathbb{L}_{=1} &= \mathrm{Lin}\{h_i\colon h_i \text{ correspond to eigenvalue } \lambda,\ |\lambda| = 1\},\\
\mathbb{L}_{>1} &= \mathrm{Lin}\{h_i\colon h_i \text{ correspond to eigenvalue } \lambda,\ |\lambda| > 1\}.
\end{aligned}$$

In [19] it is demonstrated that $\mathcal{Y}_{p,\infty}$ and $\mathcal{X}_{p,\infty}$ are bounded if and only if the following conditions are true respectively:

$$Y_\infty = (b, Ab, A^2b, \ldots) \in l^n_q \text{ or } b \in \mathbb{L}_{<1}, \tag{6}$$

$$X_\infty = (A^{-1}b, A^{-2}b, \ldots) \in l^n_q \text{ or } b \in \mathbb{L}_{>1}. \tag{7}$$

In these cases the following representations are valid:

$$\overline{\mathcal{Y}}_{p,\infty} = Y_\infty \mathcal{E}_p(\infty) \in \mathbb{K}_n, \tag{8}$$

$$\overline{\mathcal{X}}_{p,\infty} = X_\infty \mathcal{E}_p(\infty) \in \mathbb{K}_n. \tag{9}$$

According to (8) and (9), the limit sets $\mathcal{Y}_{p,\infty}$, $\mathcal{X}_{p,\infty}$ are convex, and therefore, for their description by algebraic inequalities, the Minkowski functional can be used [25, Section 3 §2 ch. III]:

$$\mu(u, \mathcal{U}) = \inf\{t > 0\colon u \in t\mathcal{U}\}.$$

Let us demonstrate the complexity of calculating the Minkowski functional of the sets (3) and (5) for an arbitrary value of the parameter p, and also give a special case when such description can be constructed.

Lemma 1. *Let $\mathbb{L}_1$, $\mathbb{L}_2$ be normed spaces, $\mathcal{U} \subset \mathbb{L}_1$ is convex and bounded set, $0 \in \mathrm{int}\,\mathcal{U}$, $B\colon \mathbb{L}_1 \to \mathbb{L}_2$ is the linear, surjective and bounded operator.*

Then

$$\mu(x, B\mathcal{U}) = \inf_{u \in B^{-1}(\{x\})} \mu(u, \mathcal{U}).$$

The proof of the Lemma 1 and all other statements is given in the Appendix.

Let us obtain the corollaries of Lemma 1, setting $\mathbb{L}_1 = l_p$, $\mathbb{L}_2 = \mathbb{R}^n$, $\mathcal{U} = \mathcal{E}_p(\infty)$. The choice of norm in the space $\mathbb{R}^n$ is unimportant, since the value of the Minkowski functional does not depend on the norm, and due to the equivalence of all norms in the finite-dimensional space [25]. Hence, the operator B is limited for any norm in $\mathbb{R}^n$. But for brevity of notation we assume that $\mathbb{R}^n$ is Euclidean with the scalar product defined by the following relation:

$$(x, y) = x^{\mathrm{T}} y = \sum_{i=1}^{n} x_i y_i.$$

Let us introduce the nonlinear operator $I_p(u)\colon l_p \to l_q$ according to the following formula:

$$I_p(u) = \left(\operatorname{sgn}(u_1)|u_1|^{p-1}, \operatorname{sgn}(u_2)|u_2|^{p-1}, \ldots\right).$$

The inverse operator to I_p is the operator I_q. Let us denote by $B^*\colon \mathbb{R}^n \to l_q$ the operator adjoint to B.

Lemma 2. *Let $B \in l_q^n$ be the surjection. Then for any $x \in \mathbb{R}^n$ it is true that*

$$\mu(x, B\mathcal{E}_p(\infty)) = \|B^*\lambda\|_{l_q}^{q-1},$$

where $\lambda \in \mathbb{R}^n$ satisfies the following condition:

$$BI_q(B^*\lambda) = x. \tag{10}$$

According to Lemma 2 and the representations (8) and (9), the calculation of the Minkowski functional for the reachable and controllable limit sets can be reduced to solving a system of nonlinear equations of the form (10) when choosing operators Y_∞ and X_∞ as B, respectively, which is a nontrivial problem in the general case. Although, when $p = q = 2$, the solution of the system can be obtained in explicit form.

Corollary 1. *Let $p = q = 2$, $B \in l_q^n$ be the surjection.*

Then for any $x \in \mathbb{R}^n$ it is true that

$$\mu(x, B\mathcal{E}_2(\infty)) = \sqrt{x^{\mathrm{T}}(BB^*)^{-1}x}.$$

The application of Corollary 1 to construct $\mathcal{Y}_{2,\infty}$ and $\mathcal{X}_{2,\infty}$ is determined by the possibility of constructing explicitly the matrix $BB^* \in \mathbb{R}^{n\times n}$, which, according to the definition of the operator B, reduces to the calculation of a convergent series:

$$BB^* = \sum_{k=1}^{\infty} b_k b_k^{\mathrm{T}},$$

where B is assumed to be equal to Y_∞ or X_∞.

Lemma 3. *Let $A \in \mathbb{R}^{n\times n}$ have n linearly independent eigenvectors $h_1, \ldots, h_n \in \mathbb{C}^n$ corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$, $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$, $S = (h_1, \ldots, h_n) \in \mathbb{C}^{n\times n}$. The following notations are used:*

$$H = S \begin{pmatrix} \beta_{11} & \ldots & \beta_{1n} \\ \vdots & \ddots & \vdots \\ \beta_{n1} & \ldots & \beta_{nn} \end{pmatrix} S^{\mathrm{T}},$$

$$\begin{pmatrix} \alpha_{11} & \ldots & \alpha_{1n} \\ \vdots & \ddots & \vdots \\ \alpha_{n1} & \ldots & \alpha_{nn} \end{pmatrix} = S^{-1} b b^{\mathrm{T}} (S^{-1})^{\mathrm{T}}, \quad \beta_{ij} = \begin{cases} \dfrac{\alpha_{ij}}{1 - \lambda_i \lambda_j}, & \lambda_i \lambda_j \neq 1, \\ 0, & \lambda_i \lambda_j = 1. \end{cases}$$

Then

1) *in the case* $b \in \mathbb{L}_{<1}$ *the following representation is true:*

$$\overline{\mathcal{Y}}_{2,\infty} = \{x \in \mathbb{R}^n \colon x^{\mathrm{T}} H_{\mathcal{Y},\infty} x \leqslant 1\},$$

where $H_{\mathcal{Y},\infty} = H^{-1}$;

2) *in the case* $b \in \mathbb{L}_{>1}$ *and* $\det A \neq 0$ *the following representation is true:*

$$\overline{\mathcal{X}}_{2,\infty} = \left\{x \in \mathbb{R}^n \colon x^{\mathrm{T}} H_{\mathcal{X},\infty} x \leqslant 1\right\},$$

where $H^{-1}_{\mathcal{X},\infty} = -H^{-1}$.

4. REACHABLE AND NULL-CONTROLLABLE LIMIT SETS AS THE FIXED-POINT

Let us present properties of the contraction mapping principle and fixed-points that are useful for representing the sets (3) and (5). It is known that the sets (2) and (4) are convex compact sets and can be represented as the image of $\mathcal{E}_p(\infty)$ under a linear mapping [19, Lemma 9]:

$$\mathcal{Y}_p(N) = Y_N \mathcal{E}_p(\infty),\ Y_N = (b, Ab, \ldots, A^{N-1}b, 0, \ldots) \in l_q^n, \tag{11}$$

$$\mathcal{X}_p(N) = X_N \mathcal{E}_p(\infty),\ X_N = (A^{-1}b, A^{-2}b, \ldots, A^{-N}b, 0, \ldots) \in l_q^n,\ \det A \neq 0. \tag{12}$$

Let us represent the operators X_∞ and Y_∞ as fixed points of the contraction mapping. To do this, we introduce two linear and bounded operators $\mathbf{MULT}_A, \mathbf{R} \colon l_q^n \to l_q^n$:

$$\mathbf{MULT}_A B' = (Ab_1, Ab_2, \ldots),\ A \in \mathbb{R}^{n\times n},$$
$$\mathbf{R} B' = (0, b_1, b_2, \ldots).$$

For arbitrary $A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$ we define the mapping $\mathbf{F}_{A,b} \colon l_q^n\ to l_q^n$ as follows:

$$\mathbf{F}_{A,b}(B') = \mathbf{R} \circ \mathbf{MULT}_A B' + (b, 0, 0, \ldots) = (b, Ab_1, Ab_2, \ldots). \tag{13}$$

For an arbitrary $M \in \mathbb{N}$ we denote by $\mathbf{F}^{(M)}_{A,b} \colon l_q^n \to l_q^n$ the M-fold composition of the mapping $\mathbf{F}_{A,b}$:

$$\mathbf{F}^{(M)}_{A,b}(B') = (\underbrace{\mathbf{F}_{A,b} \circ \ldots \circ \mathbf{F}_{A,b}}_{M})(B').$$

Lemma 4. *Let* $Y_\infty \in l_q^n$. *Then* Y_∞ *is a fixed-point of the mapping* $\mathbf{F}_{A,b}$.

Lemma 5. *Let* $\det A \neq 0$, $X_\infty \in l_q^n$. *Then* X_∞ *is a fixed-point of the mapping* $\mathbf{F}_{A^{-1},A^{-1}b}$.

Lemma 6. *Let all eigenvalues of the matrix* $A \in \mathbb{R}^{n\times n}$ *be strictly less than 1 in absolute value. Then for any* $b \in \mathbb{R}^n$

1) *there exists* $M \in \mathbb{N}$ *such that* $A^M \colon \mathbb{R}_r^n \to \mathbb{R}_r^n$ *is a contraction mapping with contraction factor* $\alpha_r \in [0; 1)$;

2) $\mathbf{F}^{(M)}_{A,b}$ *is the the contraction mapping with the contraction factor* α_r.

Corollary 2. *Let all eigenvalues of the matrix* $A \in \mathbb{R}^{n\times n}$ *be strictly less than 1 in absolute value. Then* Y_∞ *is the only fixed-point of the mapping* $\mathbf{F}_{A,b}$. *If* $M \in \mathbb{N}$ *is a number such that* $A^M \colon \mathbb{R}_r^n \to \mathbb{R}_r^n$ *is a contraction mapping with contraction factor* $\alpha_r \in [0; 1)$, *then*

$$\|Y_\infty - Y_{NM}\|_{l_q^n} \leqslant \frac{\alpha_r^N}{1-\alpha_r}\|Y_M\|_{l_q^n}.$$

Corollary 3. *Let all eigenvalues of the matrix $A \in \mathbb{R}^{n\times n}$ be strictly greater than 1 in absolute value. Then X_∞ is the only fixed-point of the mapping $\mathbf{F}_{A^{-1},A^{-1}b}$. If $M \in \mathbb{N}$ is a number such that $A^{-M}\colon \mathbb{R}^n_r \to \mathbb{R}^n_r$ is a contraction mapping with contraction factor $\beta_r \in [0;1)$, then*

$$\|X_\infty - X_{NM}\|_{l_q^n} \leqslant \frac{\beta_r^N}{1-\beta_r}\|X_M\|_{l_q^n}.$$

Estimates for the operators Y_∞ and X_∞, obtained by the simple iteration method in Corollaries 2 and 3, can also be extended to the sets (3) and (5).

Lemma 7. *Let $B', C' \in l_q^n$. Then*

$$\rho_H(B'\mathcal{E}_p(\infty), C'\mathcal{E}_p(\infty)) \leqslant \|B' - C'\|_{l_q^n}.$$

Theorem 1. *Let all eigenvalues of $A \in \mathbb{R}^{n\times n}$ be strictly less than 1 in absolute value, $M \in \mathbb{N}$ such that the mapping $A^M\colon \mathbb{R}^n_r \to \mathbb{R}^n_r$ is is contraction with contraction factor $\alpha_r \in [0;1)$. Then for any $N \in \mathbb{N}$ the following inequality holds:*

$$\rho_H\left(\overline{\mathcal{Y}}_{p,\infty}, \mathcal{Y}_p(NM)\right) \leqslant \frac{\alpha_r^N}{1-\alpha_r}\|Y_M\|_{l_q^n}.$$

Theorem 2. *Let all eigenvalues $A \in \mathbb{R}^{n\times n}$ be strictly greater than 1 in absolute value, $M \in \mathbb{N}$ such that the mapping $A^{-M}\colon \mathbb{R}^n_r \to \mathbb{R}^n_r$ is contraction with contraction factor $\beta_r \in [0;1)$. Then for any $N \in \mathbb{N}$ the following inequality holds:*

$$\rho_H\left(\overline{\mathcal{X}}_{p,\infty}, \mathcal{X}_p(NM)\right) \leqslant \frac{\beta_r^N}{1-\beta_r}\|X_M\|_{l_q^n}.$$

Corollary 2 and Theorem 1 are based on the fact that the constructed operator $\mathbf{F}^{(M)}_{A,b}$ turns out to be contractive if the matrix A^M has the similar property, and it also derives the contraction factor of this matrix. It is possible to ensure that for some $M \in \mathbb{N}$ the mapping A^M turns out to be a contraction if and only if all eigenvalues of A are strictly less than 1 in modulus. It should be noted that this condition is only a sufficient condition for the boundedness of the reachable limit set $\mathcal{Y}_{p,\infty}$, but not necessary. A necessary and sufficient condition for boundedness is the inclusion $b \in \mathbb{L}_{<1}$ [19]. Even if the matrix A has eigenvalues greater than or equal 1 in modulus, satisfying the condition $b \in \mathbb{L}_{<1}$ will lead to the boundedness of the set $\mathcal{Y}_{p,\infty}$. However, it is impossible to directly use the apparatus of contraction mappings for its construction due to the absence of a conctraction factor of matrix A^M for any $M \in \mathbb{N}$.

Nevertheless, for $b \in \mathbb{L}_{<1}$ it is possible to reduce the phase space of the system (1) to an invariant subspace $\mathbb{L}_{<1}$, on which the mapping A have only its eigenvalues that are strictly less than 1 in absolute value. This allow us to use Corollary 2 and Theorem 1 to construct the set $\mathcal{Y}_{p,\infty}$. Similar reasoning is valid for the set $\mathcal{X}_{p,\infty}$, Corollary 3 and Theorem 2 when replacing A with A^{-1}, b with $A^{-1}b$ and $\mathbb{L}_{<1}$ with $\mathbb{L}_{>1}$.

Let us separately note the case, when expanding b in a real Jordan basis A, the components corresponding to $\mathbb{L}_{=1}$ turn out to be different from 0, i.e. A has eigenvalues modulo 1. Then both sets $\mathcal{Y}_{p,\infty}$ and $\mathcal{X}_{p,\infty}$ turn out to be unbounded, which does not allow them to be represented as fixed points of contraction mappings.

5. METHOD FOR CONSTRUCTING EXTERNAL ESTIMATES OF LIMIT SETS

Let us consider the questions of constructing external estimates for the sets $\mathcal{Y}_{p,\infty}$ and $\mathcal{X}_{p,\infty}$. In [19] methods are proposed for constructing the sets $\mathcal{Y}_p(N)$ and $\mathcal{X}_p(N)$ for any arbitrary $N \in \mathbb{N}$

based on an exact description of their support functions. Moreover, Theorems 1 and 2 give an a priori estimate of the accuracy when considering the sets (2) and (4) as an internal approximation of the limit sets (3) and (5) respectively. In combination with the properties of the Hausdorff distance, this also allows us to construct an external approximation.

To do this, let us denote by $\mathcal{B}^r_R(x_0) \subset \mathbb{R}^n_r$ a ball of radius R with centers at x_0 in the space $\mathbb{R}^n_r$.

Theorem 3. *Let all eigenvalues of $A \in \mathbb{R}^{n\times n}$ be strictly less than 1 in absolute value, $M \in \mathbb{N}$ be such that the mapping $A^M\colon \mathbb{R}^n_r \to \mathbb{R}^n_r$ is contraction with contraction factor $\alpha_r \in [0;1)$. Then for any $N \in \mathbb{N}$ the following inclusion is true:*

$$\mathcal{Y}_{p,\infty} \subset \mathcal{Y}_p(NM) + \mathcal{B}^r_{R_N}(0),$$
$$R_N = \frac{\alpha_r^N}{1-\alpha_r}\|Y_M\|_{l^n_q}.$$

Theorem 4. *Let all eigenvalues $A \in \mathbb{R}^{n\times n}$ be strictly greater than 1 in absolute value, $M \in \mathbb{N}$ be such that the mapping $A^{-M}\colon \mathbb{R}^n_r \to \mathbb{R}^n_r$ is contraction with contraction factor $\beta_r \in [0;1)$. Then for any $N \in \mathbb{N}$ the following inclusion is true:*

$$\mathcal{X}_{p,\infty} \subset \mathcal{X}_p(NM) + \mathcal{B}^r_{R_N}(0),$$
$$R_N = \frac{\beta_r^N}{1-\beta_r}\|X_M\|_{l^n_q}.$$

Since the value R_N converges to 0 under the assumptions of Theorems 3 and 4, these statements make it possible to construct, with an arbitrary accuracy, external estimates of the reachable limit sets $\mathcal{Y}_{p,\infty}$ and null-controllable limit sets $\mathcal{X}_{p,\infty}$ of system (1) under the assumption that reachable sets $\{\mathcal{Y}_p(N)\}_{N=0}^{\infty}$ and null-controllable sets $\{\mathcal{X}_p(N)\}_{N=0}^{\infty}$ in a finite number of steps were constructed. To construct them, we can use the results presented in [19, Theorem 1], where for Kallman-controlled systems for the sets (2) and (4) the descriptions of an arbitrary support hyperplane and tangent points are explicitly indicated depending on the support vector.

It is somewhat difficult to calculate the values of M, α_r, β_r, $\|Y_M\|_{l^n_q}$ and $\|X_M\|_{l^n_q}$. In general, α_r is the operator norm of $A^M\colon \mathbb{R}^n_r \to \mathbb{R}^n_r$:

$$\alpha_r = \max_{\|x\|_r \leqslant 1} \|A^M x\|_r. \tag{14}$$

The convex programming problem (14) can be solved numerically for a chosen value of the parameter $r \in [1;\infty]$. Moreover, for the values $r \in \{1, 2, \infty\}$ the analytical representations are known [25]:

$$\alpha_1 = \max_{1\leqslant j\leqslant n} \sum_{i=1}^n |a_{ij}|, \quad \alpha_2 = \sqrt{\sum_{i=1}^n\sum_{j=1}^n a_{ij}^2}, \quad \alpha_\infty = \max_{1\leqslant i\leqslant n} \sum_{j=1}^n |a_{ij}|,$$

where a_{ij} denotes the components of the matrix A^M. The value of β_r is determined in a similar way by replacing the matrix A with A^{-1}.

The value of M can be determined by sequentially calculating α_r or β_r until the condition $\alpha_r \in [0;1)$ is satisfied when constructing $\mathcal{Y}_{p,\infty}$ or the condition $\beta_r \in [0;1)$ is satisfied when constructing $\mathcal{X}_{p,\infty}$. A priori estimates of M are unknown, although for the case, where A has n linearly independent eigenvectors, $M = 1$.

We present the methods for calculating $\|Y_M\|_{l^n_q}$ and $\|X_M\|_{l^n_q}$ in the form of the following theorem.

Theorem 5. *Let for some $M \in \mathbb{N}$ the following representation be true:*

$$B' = \begin{pmatrix} y^1 \\ \vdots \\ y^n \end{pmatrix} = \begin{pmatrix} b_{11} & \dots & b_{1M} & 0 & \dots \\ \vdots & & \vdots & \vdots & \\ b_{n1} & \dots & b_{nM} & 0 & \dots \end{pmatrix} \in l_q^n.$$

Then the following estimates are valid:

1) *for all $r \in [1;\infty)$ and $p > 1$, the calculation of $\|B'\|_{l_q^n}$ reduces to solving the following convex programming problem:*

$$\left(\|B'\|_{l_q^n}\right)^r = \max_{\sum\limits_{k=1}^{M} |u_k|^p = 1} \sum_{i=1}^{n} \left| \sum_{k=1}^{M} b_{ik} u_k \right|^r;$$

2) *for all $r \in [1;\infty)$ and $p > 1$*

$$\|B'\|_{l_q^n} \leqslant \left(\sum_{i=1}^{n} \left(\sum_{k=1}^{M} |b_{ik}|^q \right)^{\frac{r}{q}} \right)^{\frac{1}{r}};$$

3) *if $M = 1$, then*

$$\|B'\|_{l_q^n} = \left(\sum_{i=1}^{n} |b_{i1}|^r \right)^{\frac{1}{r}};$$

4) *if $r = p = 2$, then*

$$\|B'\|_{l_q^n} = \sqrt{\max_{\lambda \in \sigma(B^{\mathrm{T}} B)} |\lambda|},$$

$$B = \begin{pmatrix} b_{11} & \dots & b_{1M} \\ \vdots & & \vdots \\ b_{n1} & \dots & b_{nM} \end{pmatrix} \in \mathbb{R}^{n \times M},$$

$$\sigma(A) = \{\lambda \in \mathbb{C} \colon \lambda \text{ is eigenvalue of } A\};$$

5) *if $r = \infty$, then*

$$\|B'\|_{l_q^n} = \max_{i=\overline{1,n}} \left(\sum_{k=1}^{M} |b_{ik}|^q \right)^{\frac{1}{q}};$$

if $r = 1$, then

$$\|B'\|_{l_q^n} = \max_{\substack{\gamma_i \in \{-1;1\} \\ i=\overline{1,n}}} \left(\sum_{k=1}^{M} \left| \sum_{i=1}^{n} \gamma_i b_{ik} \right|^q \right)^{\frac{1}{q}}.$$

Theorem 5 allows us to use external estimates obtained in Theorems 3 and 4 for any value of the parameter r. However the most suitable values are $r = 1$ and $r = \infty$, since in these cases analytical expressions for α_r, β_r, $\|Y_M\|_{l_q^n}$ and $\|X_M\|_{l_q^n}$ are known, and the ball $\mathcal{B}_R^r(0)$ is a polyhedron:

$$\mathcal{B}_R^1(0) = \operatorname{conv}\left\{\begin{pmatrix}\pm1\\0\\\vdots\\0\end{pmatrix}, \begin{pmatrix}0\\\pm1\\\vdots\\0\end{pmatrix}, \begin{pmatrix}0\\0\\\vdots\\\pm1\end{pmatrix}\right\},$$

$$\mathcal{B}_R^\infty(0) = \operatorname{conv}\left\{\begin{pmatrix}\gamma_1\\\gamma_2\\\vdots\\\gamma_n\end{pmatrix} : \gamma_i \in \{-1;1\}, i = \overline{1,n}\right\}.$$

6. EXAMPLES

Let us demonstrate the results of Theorems 3–5 by examples.

Example 1. Let the system (1) have the following matrices for $p = \frac{4}{3}$:

$$A = 0.9\begin{pmatrix}\cos(2) & \sin(2)\\ -\sin(2) & \cos(2)\end{pmatrix}, \quad b = \begin{pmatrix}2\\2\end{pmatrix}. \tag{15}$$

Consider two values of $r \in \{1, \infty\}$. Let's put $M = 4$. Then the corresponding operator norms A^M, defined by the relations (14), take the fiollowing values:

$$\alpha_1 = \alpha_\infty = 0.4648.$$

Taking into account items 5 and 6 of Theorem 5 the following equalities are true:

$$\|Y_M\|_{l_4^n} = \begin{cases} 4.0976, & r = 1,\\ 2.4962, & r = \infty.\end{cases}$$

Let us calculate, according to Theorem 3, the value of R_N for different $N \in \{1, 2, 3, 4\}$. In the case $r = 1$ we obtain the following results:

$$R_1 = 3.5592,\ R_2 = 1.6545,\ R_3 = 0.7691,\ R_4 = 0.3575.$$

In the case $r = \infty$ we obtain the following results:

$$R_1 = 2.1681,\ R_2 = 1.0078,\ R_3 = 0.4685,\ R_4 = 0.2178.$$

Then we can construct estimates for the set $\mathcal{Y}_{\frac{4}{3},\infty}$ according to Theorem 3. The results are presented in Figs. 1 and 2. Sets $\mathcal{Y}_{\frac{4}{3}}(4)$, $\mathcal{Y}_{\frac{4}{3}}(8)$, $\mathcal{Y}_{\frac{4}{3}}(12)$, $\mathcal{Y}_{\frac{4}{3}}(16)$ are constructed by the method presented in [19]. The dotted lines indicate external estimates of the set $\mathcal{Y}_{\frac{4}{3},\infty}$ for different $N \in \{1, 2, 3, 4\}$ and for $r = 1$ in Fig. 1 and for $r = \infty$ in Fig. 2.

Example 2. Let the system (1) have the following matrices for $p = 4$:

$$A = \begin{pmatrix}0.8 & 0.2\\ 0 & 0.7\end{pmatrix}, \quad b = \begin{pmatrix}1\\2\end{pmatrix}. \tag{16}$$

Consider two values of $r \in \{1, \infty\}$. Let's put $M = 4$. Then the corresponding operator norms A^M, defined by the relations (14), take the following values:

$$\alpha_1 = 0.5791,\ \alpha_\infty = 0.7486.$$

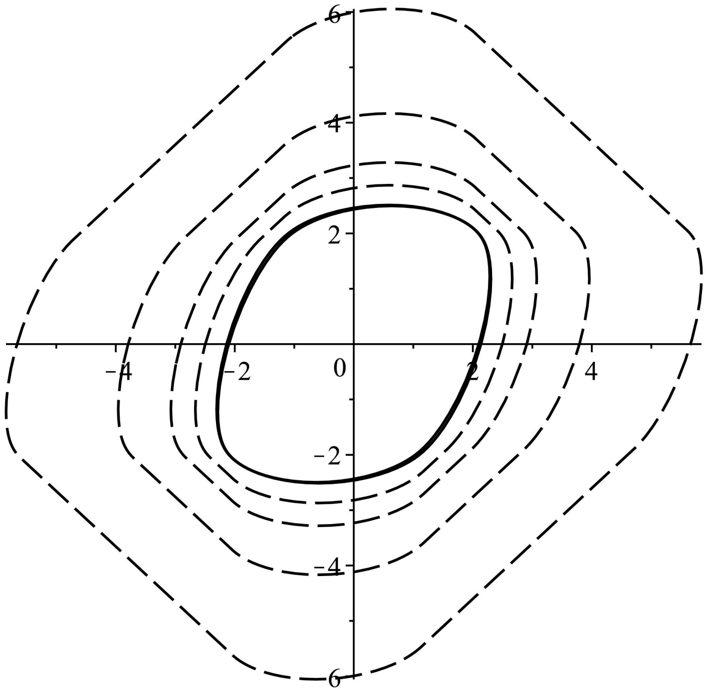

Fig. 1. Reachable sets $\mathcal{Y}_{\frac{4}{3}}(4)$, $\mathcal{Y}_{\frac{4}{3}}(8)$, $\mathcal{Y}_{\frac{4}{3}}(12)$, $\mathcal{Y}_{\frac{4}{3}}(16)$ (solid lines) and estimates $\mathcal{Y}_{\frac{4}{3},\infty}$ (dotted lines), obtained based on Theorem 3 for $r = 1$ and system matrices (15).

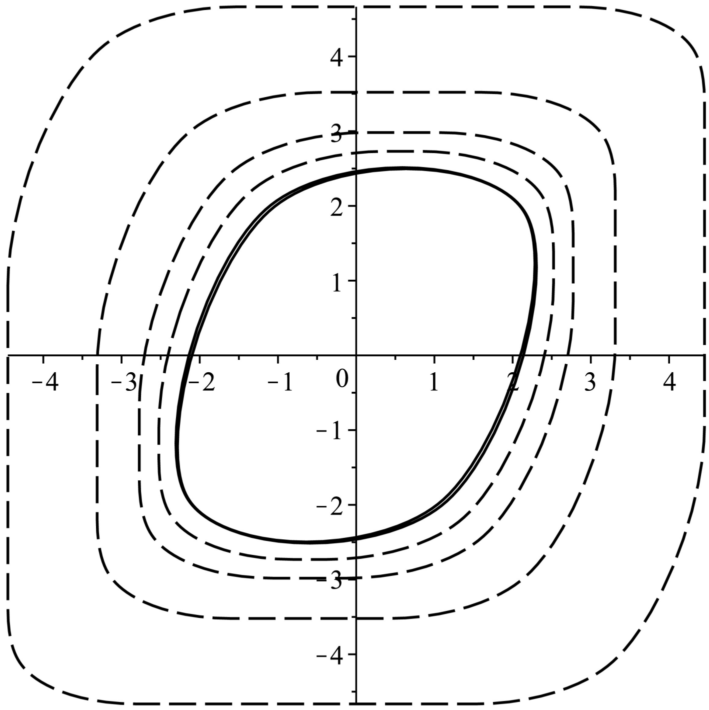

Fig. 2. Reachable sets $\mathcal{Y}_{\frac{4}{3}}(4)$, $\mathcal{Y}_{\frac{4}{3}}(8)$, $\mathcal{Y}_{\frac{4}{3}}(12)$, $\mathcal{Y}_{\frac{4}{3}}(16)$ (solid lines) and estimates $\mathcal{Y}_{\frac{4}{3},\infty}$ (dotted lines), obtained based on Theorem 3 for $r = \infty$ and system matrices (15).

Taking into account items 5 and 6 of Theorem 5 the following equalities are true:

$$\|Y_M\|_{l_4^n} = \begin{cases} 6.8891, & r = 1, \\ 3.6717, & r = \infty. \end{cases}$$

Let us calculate, according to Theorem 3, the value of R_N for different $N \in \{1, 2, 3, 4\}$. In the case $r = 1$ we obtain the following results:

$$R_1 = 9.4784,\ R_2 = 5.4890,\ R_3 = 3.1887,\ R_4 = 1.8408.$$

In the case $r = \infty$ we obtain the following results:

$$R_1 = 10.9333,\ R_2 = 8.1847,\ R_3 = 6.1271,\ R_4 = 4.5867.$$

Then we can construct estimates for the set $\mathcal{Y}_{4,\infty}$ according to Theorem 3. The results are presented in Figs. 3 and 4. The sets $\mathcal{Y}_4(4)$, $\mathcal{Y}_4(8)$, $\mathcal{Y}_4(12)$, $\mathcal{Y}_4(16)$ are constructed by the method presented in [19]. The dotted lines indicate external estimates of the set $\mathcal{Y}_{4,\infty}$ for different $N \in \{1, 2, 3, 4\}$ for $r = 1$ in Fig. 3 and for $r = \infty$ in Fig. 4.

In Example 1 the approximation accuracy turned out to be higher for $r = \infty$, while in Example 2 better results are obtained for $r = 1$. In the general case, it is possible to calculate estimates for various values of the parameter $r \in [1; \infty]$, and consider the final estimate in the form of their intersection.

Example 3. Let us consider separately the case $p = 2$, for which reachable limit sets can be constructed explicitly based on Lemma 3. Note that from the point of view of Lemma 3 it does not matter whether the eigenvalues of the matrix A are real or complex. In intermediate calculations when constructing the matrix $H_{\mathcal{Y},\infty}$, complex numbers may be used, but the resulting matrix of quadratic form, which defines the structure of $\mathcal{Y}_{2,\infty}$, will be real in any case.

For the case (15) it is true that

$$\lambda_1 = -0.3329 + 0.7274\mathbf{i},\ \lambda_2 = -0.3329 - 0.7274\mathbf{i},\ S = \begin{pmatrix} 0.7071 & 0.7071 \\ 0.7071\mathbf{i} & -0.7071\mathbf{i} \end{pmatrix},$$

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} = \begin{pmatrix} -4\mathbf{i} & 4 \\ 4 & 4\mathbf{i} \end{pmatrix},$$

$$\begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix} = \begin{pmatrix} -0.8625 - 2.5257\mathbf{i} & 11.1111 \\ 11.1111 & -0.8625 + 2.5257\mathbf{i} \end{pmatrix}.$$

From here we finally get that

$$H_{\mathcal{Y},\infty} = \begin{pmatrix} 0.1029 & -0.0217 \\ -0.0217 & 0.0881 \end{pmatrix}.$$

The results are presented in Fig. 5. The sets $\mathcal{Y}_2(2)$, $\mathcal{Y}_2(3)$, $\mathcal{Y}_2(4)$, $\mathcal{Y}_2(5)$ were constructed by the method from [19].

Similarly, for the case (16) it is true that

$$\lambda_1 = -0.8,\ \lambda_2 = -0.7,\ S = \begin{pmatrix} 1 & -0.8944 \\ 0 & 0.4472 \end{pmatrix},$$

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} = \begin{pmatrix} 25 & 22.3607 \\ 22.3607 & 20 \end{pmatrix},\ \begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix} = \begin{pmatrix} 69.4444 & 50.8197 \\ 50.8197 & 39.5197 \end{pmatrix}.$$

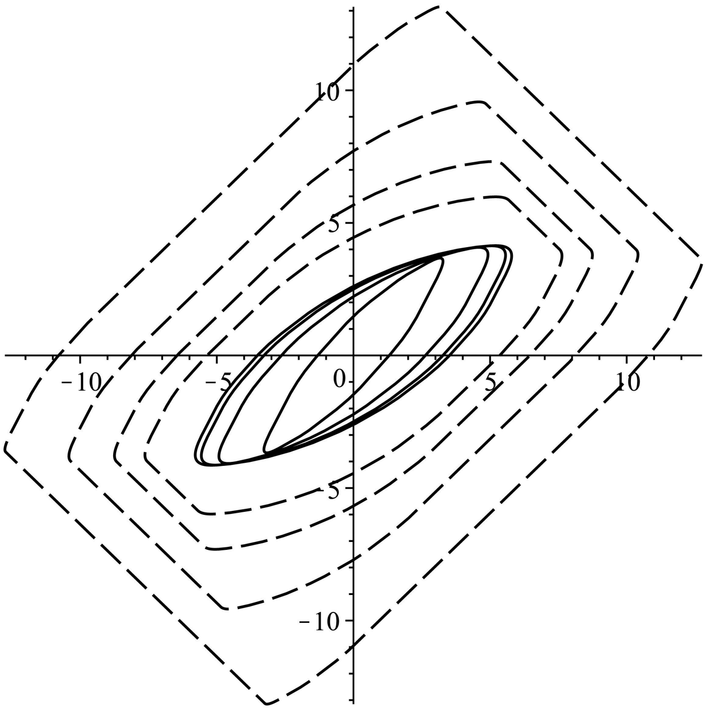

Fig. 3. Reachable sets $\mathcal{Y}_4(4)$, $\mathcal{Y}_4(8)$, $\mathcal{Y}_4(12)$, $\mathcal{Y}_4(16)$ (solid lines) and estimates $\mathcal{Y}_{4,\infty}$ (dotted lines), obtained based on Theorem 3 for $r = 1$ and system matrices (16).

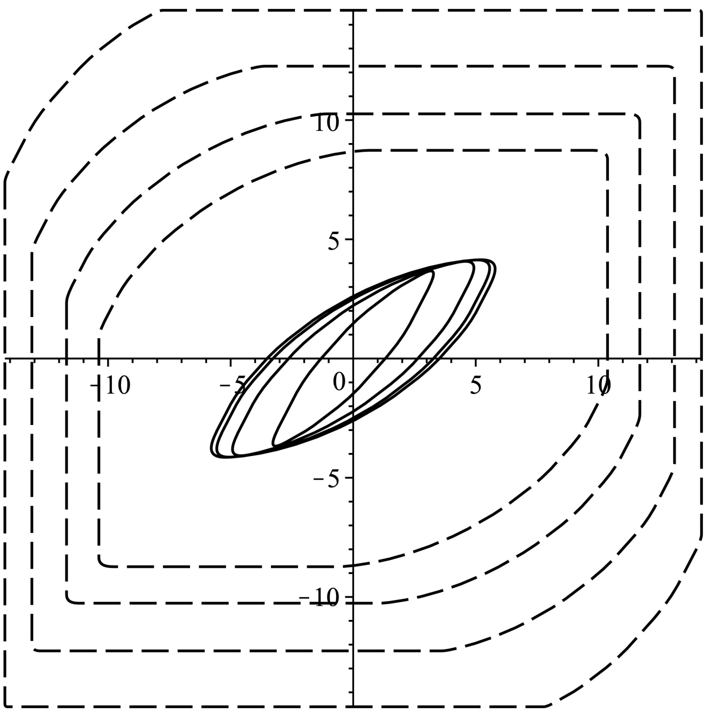

Fig. 4. Reachable sets $\mathcal{Y}_4(4)$, $\mathcal{Y}_4(8)$, $\mathcal{Y}_4(12)$, $\mathcal{Y}_4(16)$ (solid lines) and estimates $\mathcal{Y}_{4,\infty}$ (dotted lines), obtained based on Theorem 3 for $r = \infty$ and system matrices (16).

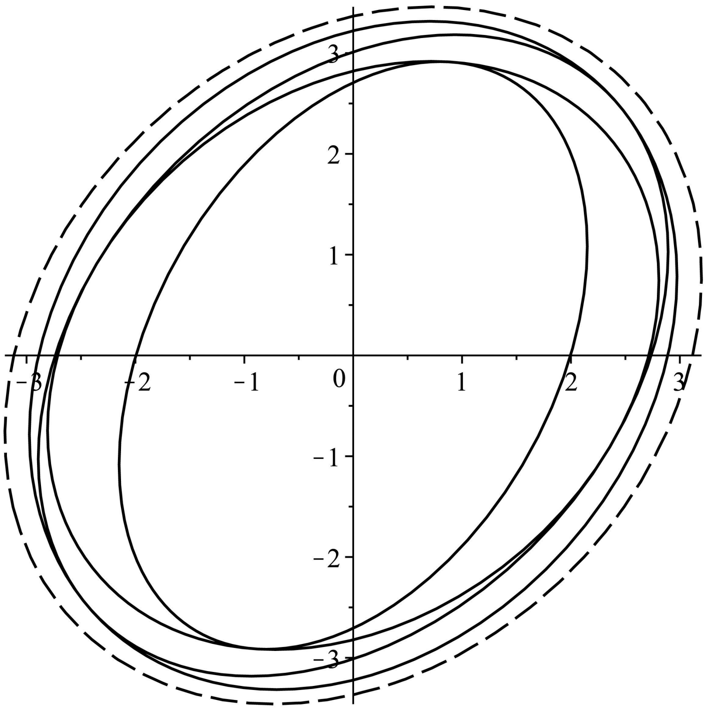

Fig. 5. Reachable sets $\mathcal{Y}_2(2)$, $\mathcal{Y}_2(3)$, $\mathcal{Y}_2(4)$, $\mathcal{Y}_2(5)$ (solid lines) and $\mathcal{Y}_{2,\infty}$ (dotted line) obtained by Lemma 3 for the matrices of the system (15).

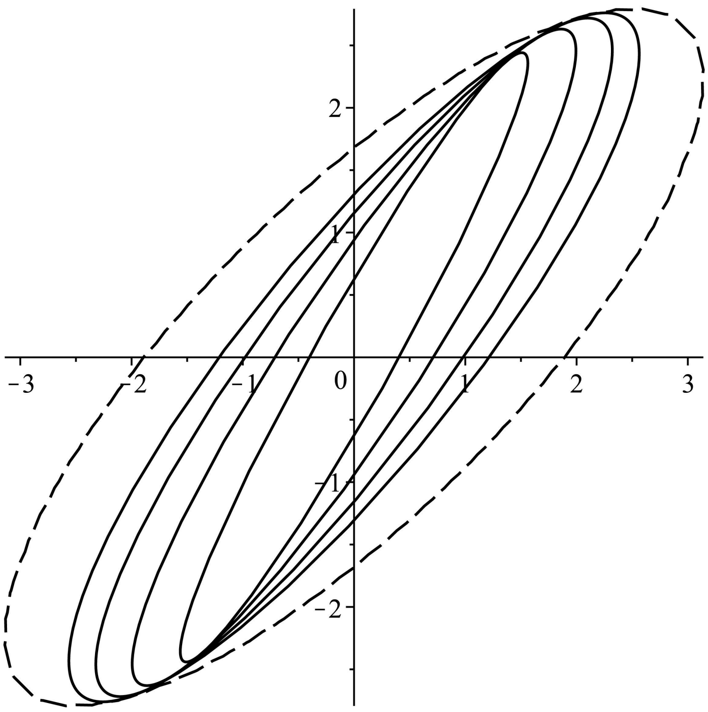

Fig. 6. Reachable sets $\mathcal{Y}_2(2)$, $\mathcal{Y}_2(3)$, $\mathcal{Y}_2(4)$, $\mathcal{Y}_2(5)$ (solid lines) and $\mathcal{Y}_{2,\infty}$ (dotted line) obtained by Lemma 3 for the matrices of the system (16).

From here we finally get that

$$H_{\mathcal{Y},\infty} = \begin{pmatrix} 0.2788 & -0.2503 \\ -0.2503 & 0.3522 \end{pmatrix}.$$

The results are presented in Fig. 6. The sets $\mathcal{Y}_2(2)$, $\mathcal{Y}_2(3)$, $\mathcal{Y}_2(4)$, $\mathcal{Y}_2(5)$ are constructed by the method from [19].

Example 4. Taking into account the relations (12) and (11) the results of Example 1 and Figs. 1 and 2 correspond to similar constructions for the null-controllable sets $\{\mathcal{X}_{\frac{4}{3}}(N)\}_{N=0}^{\infty}$ and $\mathcal{X}_{\frac{4}{3},\infty}$ when replacing A with A^{-1} and b with $A^{-1}b$:

$$A = \frac{10}{9}\begin{pmatrix} \cos(2) & -\sin(2) \\ \sin(2) & \cos(2) \end{pmatrix}, \; b = \begin{pmatrix} \frac{10}{9}(\cos(2) - \sin(2)) \\ \frac{10}{9}(\cos(2) + \sin(2)) \end{pmatrix}.$$

Also the results of the Example 2 and Figs. 3 and 4 correspond to similar constructions for the null-controllable sets $\{\mathcal{X}_4(N)\}_{N=0}^{\infty}$ and $\mathcal{X}_{4,\infty}$ when replacing A with A^{-1} and b with $A^{-1}b$:

$$A = \begin{pmatrix} \frac{5}{4} & -\frac{5}{14} \\ 0 & \frac{10}{7} \end{pmatrix}, \quad b = \begin{pmatrix} \frac{15}{28} \\ \frac{20}{7} \end{pmatrix}.$$

7. CONCLUSION

The paper considers the problem for constructing reachable and 0-controllable limit sets for stationary linear discrete-time systems with a summary constraint on scalar control. It is demonstrated that in the general case, the calculation of the Minkowski functional for given sets reduces to the operation of projecting a ball from the normed space l_p onto a finite-dimensional phase space and to solving systems of nonlinear equations. For the case of quadratic constraints and a diagonalizable matrix of the system, these equations can be solved analytically, which makes it possible to describe the Minkowski functional in explicit form. In the general case, the reachable and null-controllable limit sets can be represented as fixed points of a contraction mapping in the metric space of compacta, the contraction coefficient of which can be calculated numerically. This fact allows us to estimate the error of the simple iteration method when constructing these sets, which, in combination with the properties of the Hausdorff distance, leads to the possibility of constructing their external estimates of an arbitrary order of accuracy. In a particular case, when choosing the Minkowski or Chebyshev norm as a norm in the phase space, the resulting estimates have a polyhedral structure.

It is planned to generalize the obtained results to systems with vector control in the future. In particular, this requires developing a model for taking into account summary restrictions, for example, by using the Minkowski functional with respect to some cost set. Another problem is the construction of a recurrent description for the reachable and null-controllable sets, which would allow us to search for their limit analogues in the form of fixed points, similar to the results obtained for the case of separate control restrictions at each moment of time [15].

FUNDING

This work was supported in part by the Russian Science Foundation, project no. 23-21-00293.

APPENDIX

Proof of Lemma 1. By the definition of the Minkowski functional, for any $x \in \mathbb{L}_2$ the following relations are true:

$$\mu(x, B\mathcal{U}) = \inf\{t > 0\colon x \in tB\mathcal{U}\} = \inf\{t > 0\colon \exists u \in t\mathcal{U},\ x = Bu\}$$
$$= \inf\{t > 0\colon \exists u \in B^{-1}(\{x\}),\ u \in t\mathcal{U}\} = \inf_{u \in B^{-1}(\{x\})} \inf\{t > 0\colon u \in t\mathcal{U}\} = \inf_{u \in B^{-1}(\{x\})} \mu(u, \mathcal{U}).$$

Lemma 1 is proven.

Proof of Lemma 2. Since, due to the Riesz theorem, the operator B is linear and bounded, then according to Lemma 1

$$(\mu(x, B\mathcal{E}_p(\infty)))^p = \left(\inf_{u \in B^{-1}(\{x\})} \mu(u, \mathcal{E}_p(\infty))\right)^p = \inf_{\substack{Bu=x \\ u \in l_p}} \sum_{k=1}^{\infty} |u_k|^p. \tag{A.1}$$

To solve the optimization problem (A.1) we will use the Lagrange multiplier method for infinite-dimensional spaces [29]. The Lagrange function for $\lambda \in \mathbb{R}^n$ has the following form

$$L(u, \lambda) = \sum_{k=1}^{\infty} |u_k|^p + \lambda^{\mathrm{T}}(x - Bu).$$

Then the search for a minimum in the problem (A.1) is reduced to solving the system of equations

$$\begin{cases} \dfrac{\partial L}{\partial u_k} = 0,\ k \in \mathbb{N}, \\ Bu = x, \end{cases} \qquad \begin{cases} pI_p(u) - B^*\lambda = 0, \\ Bu = x, \end{cases} \qquad \begin{cases} u = I_p^{-1}\left(\dfrac{1}{p}B^*\lambda\right) = I_q\left(\dfrac{1}{p}B^*\lambda\right), \\ Bu = x. \end{cases}$$

Hence, taking into account (A.1) and the identity $\sum_{k=1}^{\infty} |u_k|^p = (u, I_p(u))$ it follows that

$$(\mu(x, B\mathcal{E}_p(\infty))^p = \left(I_q\left(\frac{1}{p}B^*\lambda\right), I_p\left(I_q\left(\frac{1}{p}B^*\lambda\right)\right)\right) = \left(I_q\left(\frac{1}{p}B^*\lambda\right), \frac{1}{p}B^*\lambda\right) = \left\|\frac{1}{p}B^*\lambda\right\|_{l_q}^q,$$
$$BI_q\left(\frac{1}{p}B^*\lambda\right) = x.$$

Redesignating $\frac{1}{p}\lambda$ by λ, we finally obtain

$$\mu(x, B\mathcal{E}_p(\infty)) = \|B^*\lambda\|_{l_q}^{\frac{q}{p}} = \|B^*\lambda\|_{l_q}^{q-1},$$

where $\lambda \in \mathbb{R}^n$ is determined from (10).

Lemma 2 is proven.

Proof of Corollary 1. By definition, the operator I_2 is identical. Then the condition (10) take the form

$$BB^*\lambda = x.$$

Since B is surjective, the operator $BB^*\colon \mathbb{R}^n \to \mathbb{R}^n$ and the matrix generating it are invertible, which leads to the relation

$$\lambda = (BB^*)^{-1}x.$$

Taking into account Lemma 2 we obtain

$$\mu(x, B\mathcal{E}_2(\infty)) = \|B^*(BB^*)^{-1}x\|_2 = \sqrt{(B^*(BB^*)^{-1}x, B^*(BB^*)^{-1}x)} = \sqrt{x^{\mathrm{T}}(BB^*)^{-1}x}.$$

Corollary 1 is proven.

Proof of Lemma 3. Let $B = Y_\infty$. Then, taking into account the spectral decomposition of the matrix $A = S\Lambda S^{-1}$, the following equalities are true for all $k \in \mathbb{N}$:

$$\begin{aligned} b_k b_k^{\mathrm{T}} &= A^{k-1}bb^{\mathrm{T}}(A^{k-1})^{\mathrm{T}} = S\Lambda^{k-1}S^{-1}bb^{\mathrm{T}}(S^{-1})^{\mathrm{T}}\Lambda^{k-1}S^{\mathrm{T}} \\ &= S\mathrm{diag}(\lambda_1^{k-1}, \dots, \lambda_n^{k-1}) \begin{pmatrix} \alpha_{11} & \dots & \alpha_{1n} \\ \vdots & \ddots & \vdots \\ \alpha_{n1} & \dots & \alpha_{nn} \end{pmatrix} \mathrm{diag}(\lambda_1^{k-1}, \dots, \lambda_n^{k-1}) S^{\mathrm{T}} \\ &= S \begin{pmatrix} (\lambda_1\lambda_1)^{k-1}\alpha_{11} & \dots & (\lambda_1\lambda_n)^{k-1}\alpha_{1n} \\ \vdots & \ddots & \vdots \\ (\lambda_n\lambda_1)^{k-1}\alpha_{n1} & \dots & (\lambda_n\lambda_n)^{k-1}\alpha_{nn} \end{pmatrix} S^{\mathrm{T}}. \end{aligned}$$

Due to the inclusion $b \in \mathbb{L}_{<1}$, the coefficients α_{ij} are equal to zero for all $i, j = \overline{1, n}$ such that at least one of the two eigenvalues λ_i or λ_j turns out to be greater than or equal to 1:

$$\alpha_{ij} = 0, \text{ if } |\lambda_i| \geqslant 1 \text{ or } |\lambda_j| \geqslant 1. \tag{A.2}$$

Hence, taking into account the expression for the sum of an infinite decreasing geometric progression, we get the following equality:

$$\sum_{k=1}^{\infty} (\lambda_i\lambda_j)^{k-1}\alpha_{ij} = \begin{cases} \dfrac{\alpha_{ij}}{1 - \lambda_i\lambda_j}, & |\lambda_i| < 1 \text{ and } |\lambda_j| < 1, \\ 0, & |\lambda_i| \geqslant 1 \text{ or } |\lambda_j| \geqslant 1, \end{cases}$$

which, due to the (A.2), coincides with the definition of β_{ij}.

Then the following chain of equalities is true:

$$\begin{aligned} BB^* = \sum_{k=1}^{\infty} b_k b_k^{\mathrm{T}} &= S \begin{pmatrix} \sum\limits_{k=1}^{\infty}(\lambda_1\lambda_1)^{k-1}\alpha_{11} & \dots & \sum\limits_{k=1}^{\infty}(\lambda_1\lambda_n)^{k-1}\alpha_{1n} \\ \vdots & \ddots & \vdots \\ \sum\limits_{k=1}^{\infty}(\lambda_n\lambda_1)^{k-1}\alpha_{n1} & \dots & \sum\limits_{k=1}^{\infty}(\lambda_n\lambda_n)^{k-1}\alpha_{nn} \end{pmatrix} S^{\mathrm{T}} = \\ &= S \begin{pmatrix} \beta_{11} & \dots & \beta_{1n} \\ \vdots & \ddots & \vdots \\ \beta_{n1} & \dots & \beta_{nn} \end{pmatrix} S^{\mathrm{T}} = H. \end{aligned}$$

This, taking into account Corollary 1, implies the equality $H_{\mathcal{Y},\infty} = H^{-1}$.

The second item of Lemma 3 is proven in a similar way under the redesignation $B = X_\infty$.

Lemma 3 is proven.

Proof of Lemma 4. The proof follows directly from (6) and (13).

Proof of Lemma 5. The proof follows directly from (7) and (13).

Proof of Lemma 6. Since all eigenvalues of the matrix A are strictly less than 1 in absolute value, according to [22, Theorem 5.6.12] the relation $\|A^k\| \overset{k\to\infty}{\longrightarrow} 0$ is true. Then, by definition of the limit there is $M \in \mathbb{N}$ for $\alpha_r \in [0;1)$ such that $\|A^M\| = \sup_{\|x\|_r \leqslant 1} \|A^M x\|_r < \alpha_r$. Since the inequality

$$\|A^M(x-y)\|_r \leqslant \|A^M\|\|x-y\|_r \leqslant \alpha_r \|x-y\|_r$$

is true, A^M is a contraction with the contraction factor $\alpha_r \in [0;1)$.

Let $B' = (b_1, b_2, \ldots) \in l_q^n$, $C' = (c_1, c_2, \ldots) \in l_q^n$. Then

$$\begin{aligned}
\left\|\mathbf{F}_{A,b}^{(M)}(B') - \mathbf{F}_{A,b}^{(M)}(C')\right\|_{l_q^n} &= \Big\|(b, Ab, \ldots A^{M-1}b, A^M b_1, A^M b_2, \ldots) \\
&\quad - (b, Ab, \ldots A^{M-1}b, A^M c_1, A^M c_2, \ldots)\Big\|_{l_q^n} \\
&= \left\|(0, \ldots, 0, A^M(b_1 - c_1), A^M(b_2 - c_2), \ldots)\right\|_{l_q^n} \\
&= \left\|(A^M(b_1 - c_1), A^M(b_2 - c_2), \ldots)\right\|_{l_q^n} \\
&= \sup_{u \in \mathcal{E}_p(\infty)} \|(A^M(b_1 - c_1), A^M(b_2 - c_2), \ldots)u\|_r \\
= \sup_{u \in \mathcal{E}_p(\infty)} \left\|\sum_{k=1}^{\infty} A^M(b_k - c_k)u_k\right\|_r &= \sup_{u \in \mathcal{E}_p(\infty)} \left\|A^M \sum_{k=1}^{\infty} b_k u_k - A^M \sum_{k=1}^{\infty} c_k u_k\right\|_r \\
&\leqslant \sup_{u \in \mathcal{E}_p(\infty)} \alpha_r \left\|\sum_{k=1}^{\infty} b_k u_k - \sum_{k=1}^{\infty} c_k u_k\right\|_r = \alpha_r \|B' - C'\|_{l_q^n}.
\end{aligned}$$

Lemma 6 is proven.

Proof of Corollary 2. The mapping $\mathbf{F}_{A,b}^{(M)}$ is a contraction due to Theorem 6, and the space l_q^n is complete. Then, by virtue of the Banach contraction mapping principle [25], there is a unique fixed-point of $\mathbf{F}_{A,b}^{(M)}$. Moreover, by construction, a fixed-point of the mapping $\mathbf{F}_{A,b}$ must also be a fixed-point of the mapping $\mathbf{F}_{A,b}^{(M)}$. Taking into account Lemma 4 the unique fixed-point of $\mathbf{F}_{A,b}^{(M)}$ is $Y_\infty \in l_q^n$. Hence, Y_∞ is the unique fixed-point of $\mathbf{F}_{A,b}$.

Note that the representation $\mathbf{F}_{A,b}^{(NM)}(O) = Y_{NM}$ is valid, where by $O\colon l_p \to \mathbb{R}_r^n$ denotes the zero operator, which is identified with the zero sequence $(0, 0, \ldots) \in l_q^n$. Then according to Theorem 6

$$\begin{aligned}
\|Y_\infty - Y_{NM}\|_{l_q^n} &= \left\|\mathbf{F}_{A,b}^{(M)}(Y_\infty) - \mathbf{F}_{A,b}^{(M)}\left(\mathbf{F}_{A,b}^{(NM-M)}(O)\right)\right\|_{l_q^n} \\
&\leqslant \alpha_r \left\|Y_\infty - \mathbf{F}_{A,b}^{(NM-M)}(O)\right\|_{l_q^n} \\
\leqslant \alpha_r \left\|Y_\infty - \mathbf{F}_{A,b}^{(NM)}(O)\right\|_{l_q^n} &+ \alpha_r \left\|\mathbf{F}_{A,b}^{(NM)}(O) - \mathbf{F}_{A,b}^{(NM-M)}(O)\right\|_{l_q^n} \\
&\leqslant \alpha_r \left\|Y_\infty - \mathbf{F}_{A,b}^{(NM)}(O)\right\|_{l_q^n} + \alpha_r^N \left\|\mathbf{F}_{A,b}^{(M)}(O) - O\right\|_{l_q^n} \\
&= \alpha_r \|Y_\infty - Y_{NM}\|_{l_q^n} + \alpha_r^N \|Y_M\|_{l_q^n},
\end{aligned}$$

$$\|Y_\infty - Y_{NM}\|_{l_q^n} \leqslant \frac{\alpha_r^N}{1 - \alpha_r} \|Y_M\|_{l_q^n}.$$

Corollary 2 is proven.

Proof of Corollary 3. The proof follows from Corollary 2 by replacing A with A^{-1}, b with $A^{-1}b$ in conjunction with Lemma 5 and the fact that the eigenvalues of A^{-1} are inverse of the eigenvalues of A [28].

Proof of Lemma 7. Let's consider the value

$$\rho_H(B'\mathcal{E}_p(\infty), C'\mathcal{E}_p(\infty))$$

$$= \max\left\{\sup_{x\in B'\mathcal{E}_p(\infty)}\inf_{y\in C'\mathcal{E}_p(\infty)}\|x-y\|_r;\ \sup_{x\in C'\mathcal{E}_p(\infty)}\inf_{y\in B'\mathcal{E}_p(\infty)}\|x-y\|_r\right\}$$

$$= \max\left\{\sup_{u\in\mathcal{E}_p(\infty)}\inf_{v\in\mathcal{E}_p(\infty)}\|B'u-C'v\|_r;\ \sup_{u\in\mathcal{E}_p(\infty)}\inf_{v\in\mathcal{E}_p(\infty)}\|C'u-B'v\|_r\right\},$$

$$\sup_{u\in\mathcal{E}_p(\infty)}\inf_{v\in\mathcal{E}_p(\infty)}\|C'u-B'v\|_r = \sup_{u\in\mathcal{E}_p(\infty)}\inf_{v\in\mathcal{E}_p(\infty)}\|B'u-C'v\|_r$$

$$= \sup_{u\in\mathcal{E}_p(\infty)}\inf_{v\in\mathcal{E}_p(\infty)}\|B'u-C'u+C'u-C'v\|_r$$

$$\leqslant \sup_{u\in\mathcal{E}_p(\infty)}\inf_{v\in\mathcal{E}_p(\infty)}(\|B'u-C'u\|_r+\|C'u-C'v\|_r)$$

$$= \sup_{u\in\mathcal{E}_p(\infty)}\left(\|(B'-C')u\|_r+\inf_{v\in\mathcal{E}_p(\infty)}\|C'(u-v)\|_r\right)$$

$$= \sup_{u\in\mathcal{E}_p(\infty)}\|(B'-C')u\|_r = \|B'-C'\|_{l_q^n}.$$

Finally we get that

$$\rho_H(B'\mathcal{E}_p(\infty), C'\mathcal{E}_p(\infty)) \leqslant \|B'-C'\|_{l_q^n}.$$

Lemma 7 is proven.

Proof of Theorem 1. The proof follows directly from Corollary 2, the representations (11) and (8), and Lemma 7.

Proof of Theorem 2. The proof follows directly from Corollary 3, the representations (12) and (9), and Lemma 7.

Proof of Theorem 3. As is known [27], for any $\mathcal{X},\mathcal{Y}\in\mathbb{K}_n$ satisfying the condition $\rho_H(\mathcal{X},\mathcal{Y})\leqslant R$, the following inclusion is true:

$$\mathcal{X}\subset\mathcal{Y}+\mathcal{B}_R^r(0).$$

From here, taking into account Theorem 1, Theorem 3 follows.

Proof of Theorem 4. The proof is similar to the proof of Theorem 3, replacing Theorem 1 with Theorem 2.

Proof of Theorem 5. 1. Item 1 follows from the definition of the operator norm and the fact that the maximum of a convex function is achieved on the boundary of the convex set [22]:

$$\|B'\|_{l_q^n} = \sup_{u\in\mathcal{E}_p(\infty)}\|B'u\|_r = \sup_{\sum_{k=1}^{\infty}|u_k|^p=1}\left(\sum_{i=1}^{n}\left|\sum_{k=1}^{\infty}b_{ik}u_k\right|^r\right)^{\frac{1}{r}}$$

$$= \left(\sup_{\sum_{k=1}^{\infty}|u_k|^p=1}\sum_{i=1}^{n}\left|\sum_{k=1}^{M}b_{ik}u_k\right|^r\right)^{\frac{1}{r}} = \left(\max_{\sum_{k=1}^{M}|u_k|^p=1}\sum_{i=1}^{n}\left|\sum_{k=1}^{M}b_{ik}u_k\right|^r\right)^{\frac{1}{r}}.$$

2. By virtue of Hölder's inequality, item 2 follows from item 1:

$$\|B'\|_{l_q^n} \leqslant \left(\max_{\sum\limits_{k=1}^{M} |u_k|^p = 1} \sum_{i=1}^{n} \left[\left(\sum_{k=1}^{M} |b_{ik}|^q \right)^{\frac{1}{q}} \left(\sum_{k=1}^{M} |u_k|^p \right)^{\frac{1}{p}} \right]^r \right)^{\frac{1}{r}} = \left(\sum_{i=1}^{n} \left(\sum_{k=1}^{M} |b_{ik}|^q \right)^{\frac{r}{q}} \right)^{\frac{1}{r}}.$$

3. Also, for $M = 1$, item 3 follows from item 1:

$$\|B'\|_{l_q^n} = \left(\max_{|u_1|^p = 1} \sum_{i=1}^{n} |b_{i1} u_1|^r \right)^{\frac{1}{r}} = \left(\sum_{i=1}^{n} |b_{i1}|^r \right)^{\frac{1}{r}}.$$

4. For $r = p = 2$ the operator norm can be represented in terms of the scalar product in $\mathbb{R}_2^n$:

$$(x, y) = \sum_{i=1}^{n} x_i y_i.$$

Then, taking into account item 1, the following representation is true:

$$\Big(\|B'\|_{l_q^n} \Big)^2 = \max_{\substack{u \in \mathbb{R}^M: \\ (u,u)=1}} (Bu, Bu) = \max_{\substack{u \in \mathbb{R}^M: \\ (u,u)=1}} (u, B^{\mathrm{T}} B u).$$

Due to the Lagrange multiplier method [29], the maximum point of the optimization problem under consideration $u^* \in \mathbb{R}^M$ satisfies the following conditions:

$$\begin{cases} \nabla \Big((u, B^{\mathrm{T}} B u) + \lambda (1 - (u, u)) \Big) = 0, \\ (u, u) = 1, \end{cases} \quad \begin{cases} 2B^{\mathrm{T}} B - 2\lambda u = 0, \\ (u, u) = 1, \end{cases} \quad \begin{cases} (B^{\mathrm{T}} B - \lambda I) u = 0, \\ (u, u) = 1. \end{cases}$$

Then, by definition, u^* is a normed eigenvector of the matrix $B^{\mathrm{T}} B$ corresponding to the eigenvalue λ^*, i.e.

$$\Big(\|B'\|_{l_q^n} \Big)^2 = (u^*, B^{\mathrm{T}} B u^*) = (u^*, \lambda^* u^*) = \lambda^*.$$

Item 4 is completely proven.

5. Item 5 follows from the representation of the operator norm B' and the Riesz theorem on the norm of a linear and bounded functional in l_p [25]:

$$\|B'\|_{l_q^n} = \sup_{u \in \mathcal{E}_p(\infty)} \max_{i=\overline{1,n}} \left| \sum_{k=1}^{\infty} b_{ik} u_k \right| = \max_{i=\overline{1,n}} \left(\sum_{k=1}^{\infty} |b_{ik}|^q \right)^{\frac{1}{q}} = \max_{i=\overline{1,n}} \left(\sum_{k=1}^{M} |b_{ik}|^q \right)^{\frac{1}{q}}.$$

6. To prove item 6, we take into account the representation $|\gamma| = \max\{\gamma, -\gamma\}$ for any $\gamma \in \mathbb{R}$ and consider the following chain of equalities:

$$\|B'\|_{l_q^n} = \sup_{u \in \mathcal{E}_p(\infty)} \sum_{i=1}^{n} \left| \sum_{k=1}^{\infty} b_{ik} u_k \right| = \sup_{u \in \mathcal{E}_p(\infty)} \sum_{i=1}^{n} \max_{\gamma_i \in \{-1;1\}} \left(\gamma_i \sum_{k=1}^{\infty} b_{ik} u_k \right)$$

$$= \max_{\substack{\gamma_i \in \{-1;1\} \\ i=\overline{1,n}}} \sup_{u \in \mathcal{E}_p(\infty)} \left| \sum_{k=1}^{\infty} \left(\sum_{i=1}^{n} \gamma_i b_{ik} \right) u_k \right| = \max_{\substack{\gamma_i \in \{-1;1\} \\ i=\overline{1,n}}} \left(\sum_{k=1}^{\infty} \left| \sum_{i=1}^{n} \gamma_i b_{ik} \right|^q \right)^{\frac{1}{q}}$$

$$= \max_{\substack{\gamma_i \in \{-1;1\} \\ i=\overline{1,n}}} \left(\sum_{k=1}^{M} \left| \sum_{i=1}^{n} \gamma_i b_{ik} \right|^q \right)^{\frac{1}{q}}.$$

Theorem 5 is completely proven.

REFERENCES

1. Propoi, A.I., *Elementy teorii optimal'nykh diskretnykh protsessov* (Elements of the Theory of Optimal Discrete Processes), Moscow: Nauka, 1973.

2. Ibragimov, D.N. and Sirotin, A.N., On the Problem of Optimal Speed for the Discrete Linear System with Bounded Scalar Control on the Basis of 0-controllability Sets, *Autom. Remote Control*, 2015, vol. 76, no. 9, pp. 1517–1540. https://doi.org/10.1134/S0005117919030019

3. Ibragimov, D.N., On the Optimal Speed Problem for the Class of Linear Autonomous Infinite-Dimensional Discrete-Time Systems with Bounded Control and Degenerate Operator, *Autom. Remote Control*, 2019, vol. 80, no. 3, pp. 393–412. https://doi.org/10.1134/S0005117919030019

4. Zaitseva, M.V. and Tochilin P.A., Constructing Estimates of Reachability Sets in Crowd Flows Modeling, *Computational Mathematics and Mathematical Physics*, 2023, vol. 63, no. 8, pp. 1527–1539. https://doi.org/10.1134/S0965542523070163

5. Tochilin, P.A., On the Construction of Nonconvex Approximations to Reach Sets of Piecewise Linear Systems, *Differential Equations*, 2015, vol. 51, no. 11, pp. 1499–1511. https://doi.org/10.1134/S0012266115110117

6. Kuntsevich, V.M. and Kurzhanski, A.B., Attainability Domains for Linear and Some Classes of Nonlinear Discrete Systems and Their Control *J. Autom. Inform. Sci.*, 2010, vol. 42, no. 1, pp. 1–18. https://doi.org/10.1615/JAutomatInfScien.v42.i1.10

7. Sirotin, A.N. and Formal'skii, A.M., Reachability and Controllability of Discrete-Time Systems under Control Actions Bounded in Magnitude and Norm, *Autom. Remote Control*, 2003, vol. 64, no. 12, pp. 1844–1857. https://doi.org/10.1023/B:AURC.0000008423.93495.be

8. Colonius, F., Cossich, J.A.N., and Santana, A.J., Controllability Properties and Invariance Pressure for Linear Discrete-Time Systems, *J. Dynam. Different. Equat.*, 2022, vol. 34, pp. 5–22. https://doi.org/10.1007/s10884-021-09966-4

9. Darup, M.S. and Monnigmann, M., On General Relations between Null-Controllable and Controlled Invariant Sets for Linear Constrained Systems, *53rd IEEE Conference on Decision and Control*, Los Angeles, 2014, pp. 6323–6328. https://doi.org/10.1109/CDC.2014.7040380

10. Ge, S.S., Zhendong, S., and Lee, T.H., Reachability and Controllability of Switched Linear Siscrete-Time Systems, *IEEE Transact. Autom. Control*, 2001, vol. 46, no. 9, pp. 1437–1441. https://doi.org/10.1109/9.948473

11. Heemels, W.P.M.H. and Camlibel, M.K., Null Controllability of Discrete-Time Linear Systems with Input and State Constraints *Proc. 47th IEEE Conf. Decis. Control.*, 2008, pp. 3487–3492. https://doi.org/10.1109/CDC.2008.4739333

12. Kaba, M.D. and Camlibel, M.K., A Spectral Characterization of Controllability for Linear Discrete-Time Systems with Conic Constraints, *SIAM J. Control Optim.*, 2015, vol. 53, no. 4, pp. 2350–2372. https://doi.org/10.1137/140960967

13. Benvenuti, L. and Farina, L., The Geometry of the Reachability Set for Linear Discrete-Time Systems with Positive Controls, *SIAM J. Matrix Anal. Appl.*, 2006, vol. 28, no. 2, pp. 306–325. https://doi.org/10.1137/040612531

14. Fucheng, L., Mengyuan, S., and Usman, Optimal Preview Control for Linear Discrete-Time Periodic Systems, *Mathematical Problems in Engineering*, 2019, pp. 1–11. https://doi.org/10.1155/2019/8434293

15. Berendakova, A.V. and Ibragimov, D.N., About the Method for Constructing External Estimates of the Limit 0-Controllability Set for the Linear Discrete-Time System with Bounded Control, *Autom. Remote Control*, 2023, vol. 84, no. 2, pp. 97–120. https://doi.org/10.25728/arcRAS.2023.95.43.001

16. Kostousova, E.K., On External Polyhedral Estimation of the Reachability Sets in the Extended Space for Linear Multistep Systems with Integral Control Constraints, *Vychisl. Tekhnol.*, 2004, vol. 9, no. 4, pp. 54–72.

17. Gayek, J.E. and Fisher, M.E., Approximating Reachable Sets for n-Dimensional Linear Discrete Systems, *IMA J. Math. Control Inform.*, 1987, vol. 4, no. 2, pp. 149–160. https://doi.org/10.1093/imamci/4.2.149

18. Ibragimov, D.N., Osokin, A.V., Sirotin, A.N., and Sypalo, K.I., On the Properties of the Limit Control Sets for a Class of Unstable Linear Systems with Discrete Time and l_1-Restrictions, *J. Comput. Syst. Sci. Int.*, 2022, vol. 61, no. 4, pp. 467–484. https://doi.org/10.1134/S1064230722040104

19. Ibragimov, D.N. and Sirotin, A.N., On Some Properties of Sets of Bounded Controllability for Stationary Linear Discrete Systems with Total Control Constraints, *J. Comput. Syst. Sci. Int.*, 2023, vol. 62, no. 6, pp. 3–32. https://doi.org/10.1134/S1064230723050088

20. Barr, A.H., Superquadrics and Angle-Preserving Transformations *IEEE-CGA*, 1981, vol. 1, no. 1, pp. 11–23. https://doi.org/10.1109/MCG.1981.1673799

21. Tobler, W.R., The Hyperelliptical and Other New Pseudo Cylindrical Equal Area Map Projections, *J. Geophys. Research*, 1973, vol. 78, no. 11, pp. 1753–1759. https://doi.org/10.1029/JB078i011p01753

22. Rockafellar, R., *Vypuklyi analiz* (Convex Analysis), Moscow: Mir, 1973.

23. Polovinkin, E.S. and Balashov, M.V., *Elementy vypuklogo i sil'no vypuklogo analiza* (Elements of Convex and Strongly Convex Analysis), Moscow: Fizmatlit, 2004.

24. Boyd, S. and Vandenberghe, L., Convex Optimization, Cambridge: Cambridge university press, 2004. https://doi.org/10.1017/CBO9780511804441

25. Kolmogorov, A.N. and Fomin, S.V., *Elementy teorii funktsii i funktsional'nogo analiza* (Elements of the Theory of Functions and Functional Analysis), Moscow: Fizmatlit, 2012.

26. Dunford, N. and Schwartz, J.T., *Linear Operators, Part 2: Spectral Theory, Self Adjoint Operators in Hilbert Space*, New York: Interscience Publishers, 1963. Translated under the title *Lineinye operatory. T. 2. Spektral'naya teoriya. Samosopryazhennye operatory v gil'bertovom prostranstve*, Moscow: Mir, 1966.

27. Kronover, R.M., *Fraktaly i khaos v dinamicheskikh sistemakh* (Fractals and Chaos in Dynamical Systems), Moscow: Postmarket, 2000.

28. Horn, R. and Johnson, C., *Matrichnyi analiz* (Matrix Analysis), Moscow: Mir, 1989.

29. Alekseev, V.M., Tihomirov, V.M., and Fomin, S.V., *Optimal'noye upravleniye* (Optimal Control), Moscow: Nauka, 1979.

This paper was recommended for publication by M.V. Khlebnikov, a member of the Editorial Board